

The Quantum Action Principle in the framework of Causal Perturbation Theory

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Abstract. In perturbative quantum field theory the maintenance of classical symmetries is quite often investigated by means of algebraic renormalization, which is based on the Quantum Action Principle. We formulate and prove this principle in a new framework, in causal perturbation theory with localized interactions. Throughout this work a universal formulation of symmetries is used: the Master Ward Identity.

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1. Introduction

The main problem in perturbative renormalization is to prove that symmetries of the underlying classical theory can be maintained in the process of renormalization. In traditional renormalization theory this is done by 'algebraic renormalization' [26]. This method relies on the '*Quantum Action Principle*' (QAP), which is due to Lowenstein [23] and Lam [22]. This principle states that the most general violation of an identity expressing a relevant symmetry ('Ward identity') can be expressed by the insertion of a local field with appropriately bounded mass dimension. Proceeding in a proper field formalism¹ by induction on the order of \hbar , this knowledge about the structure of violations of Ward identities and often cohomological results are used to remove these violations by finite renormalizations. For example, this method has been used to prove BRST-symmetry of Yang-Mills gauge theories [2, 3, 31, 17, 1].

Traditionally, algebraic renormalization is formulated in terms of a renormalization method in which the interaction is not localized (i.e. $S_{\text{int}} = \int dx \mathcal{L}_{\text{int}}(x)$,

¹By 'proper field formalism' we mean the description of a perturbative QFT in terms of the generating functional of the 1-particle irreducible diagrams.

where \mathcal{L}_{int} is a polynomial in the basic fields with constant coefficients), for example the BPHZ momentum space subtraction procedure [32, 23, 22] or the pole subtractions of dimensionally regularized integrals [5]. In [25] it is pointed out (without proof) that the QAP is a general theorem in perturbative QFT for non-localized interactions, i.e. it holds in any renormalization scheme.²

However, for the generalization of perturbative QFT to general globally hyperbolic *curved spacetimes*, it is advantageous to work with *localized interactions* (i.e. $S_{\text{int}} = \int dx \sum_{n \geq 1} (g(x))^n \mathcal{L}_{\text{int},n}(x)$, where g is a test function with compact support) and to use a renormalization method which proceeds in configuration space and in which the locality and causality of perturbative QFT is clearly visible [8, 18, 19]. It is *causal perturbation theory* (CPT) [4, 15, 14] which is distinguished by these criteria.

Since it is the framework of *algebraic QFT* [16] in which the problems specific for curved spacetimes (which mainly rely on the absence of translation invariance) can best be treated, our main goal is the perturbative construction of the net of local algebras of interacting fields ('perturbative algebraic QFT'). Using the formulation of causality in CPT, it was possible to show that for this construction it is sufficient to work with *localized* interactions [8, 12]. Hence, a main argument against localized interactions, namely that a space or time dependence of the coupling constants has not been observed in experiments, does not concern perturbative algebraic QFT. Because of the localization of the interactions, the construction of the local algebras of interacting fields is not plagued by infrared divergences, the latter appear only in the construction of physical states.

Due to these facts it is desirable to transfer the techniques of algebraic renormalization to CPT, that is to formulate the \hbar -expansion, a proper field formalism and the QAP in the framework of CPT. For the \hbar -expansion the difficulty is that CPT is a construction of the perturbation series by induction on the coupling constant, a problem solved in [11, 12]. A formulation of the QAP in the framework of CPT has partially been given in [11] and in [27]; but for symmetries relying on a variation of the fields (as e.g. BRST-symmetry) an appropriate formulation and a proof were missing up to the appearance of the paper [6]. In the latter, also a proper field formalism and algebraic renormalization are developed in the framework of CPT.

In this paper we concisely review main results of that work [6], putting the focus on the QAP. To be closer to the conventional treatment of perturbative QFT in Minkowski space and to simplify the formalism, we work with the Wightman 2-point function instead of a Hadamard function.³ Compared with [6], we formulate some topics alternatively, in particular we introduce the proper field formalism

²Causal perturbation theory, with the adiabatic limit carried out, is included in that statement.

³In [6] smoothness in the mass m is required for $m \geq 0$ which excludes the Wightman 2-point function.

without using arguments relying on Wick's theorem and the corresponding diagrammatic interpretation. In addition we prove a somewhat stronger version of the QAP.

The validity of the QAP is very general. Therefore, we investigate a universal formulation of Ward identities: the *Master Ward Identity* (MWI) [9, 13]. This identity can be derived in the framework of classical field theory simply from the fact that classical fields can be multiplied pointwise. Since this is impossible for quantum fields (due to their distributional character), the MWI is a highly non-trivial renormalization condition, which cannot be fulfilled in general, the well known anomalies of perturbative QFT are the obstructions.

2. The off-shell Master Ward Identity in classical field theory

For algebraic renormalization it is of crucial importance that the considered Ward identities hold true in classical field theory. Therefore, in this section, we derive the off-shell MWI in the classical framework. The formalism of classical field theory, which we are going to introduce, will be used also in perturbative QFT, since the latter will be obtained by deformation of the classical Poisson algebra (Sect. 3) [11, 12, 13, 14].

For simplicity we study the model of a real scalar field φ on d dimensional Minkowski space \mathbb{M} , $d > 2$. The field φ and partial derivatives $\partial^a \varphi$ ($a \in \mathbb{N}_0^d$) are evaluation functionals on the configuration space $\mathcal{C} \equiv \mathcal{C}^\infty(\mathbb{M}, \mathbb{R})$: $(\partial^a \varphi)(x)(h) = \partial^a h(x)$. Let \mathcal{F} be the space of all functionals

$$F(\varphi) : \mathcal{C} \longrightarrow \mathbb{C}, \quad F(\varphi)(h) = F(h), \quad (2.1)$$

which are localized polynomials in φ :

$$F(\varphi) = \sum_{n=0}^N \int dx_1 \dots dx_n \varphi(x_1) \dots \varphi(x_n) f_n(x_1, \dots, x_n), \quad (2.2)$$

where $N < \infty$ and the f_n 's are \mathbb{C} -valued distributions with compact support, which are symmetric under permutations of the arguments and whose wave front sets satisfy the condition

$$\text{WF}(f_n) \cap (\mathbb{M}^n \times (\overline{V}_+^n \cup \overline{V}_-^n)) = \emptyset \quad (2.3)$$

and $f_0 \in \mathbb{C}$. (\overline{V}_\pm denotes the closure of the forward/backward light-cone.) Endowed with the *classical product* $(F_1 \cdot F_2)(h) := F_1(h) \cdot F_2(h)$, the space \mathcal{F} becomes a commutative algebra. By the support of a functional $F \in \mathcal{F}$ we mean the support of $\frac{\delta F}{\delta \varphi}$.

The space of *local functionals* $\mathcal{F}_{\text{loc}} \subset \mathcal{F}$ is defined as

$$\mathcal{F}_{\text{loc}} \stackrel{\text{def}}{=} \left\{ \int dx \sum_{i=1}^N A_i(x) h_i(x) \equiv \sum_{i=1}^N A_i(h_i) \mid A_i \in \mathcal{P}, h_i \in \mathcal{D}(\mathbb{M}) \right\}, \quad (2.4)$$

where \mathcal{P} is the linear space of all polynomials of the field φ and its partial derivatives:

$$\mathcal{P} := \bigvee \left\{ \partial^a \varphi \mid a \in \mathbb{N}_0^d \right\}. \quad (2.5)$$

We consider action functionals of the form $S_{\text{tot}} = S_0 + \lambda S$ where $S_0 \stackrel{\text{def}}{=} \int dx \frac{1}{2} (\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2)$ is the free action, λ a real parameter and $S \in \mathcal{F}$ some compactly supported interaction, which may be *non-local*. The retarded Green function $\Delta_{S_{\text{tot}}}^{\text{ret}}$ corresponding to the action S_{tot} , is defined by

$$\int dy \Delta_{S_{\text{tot}}}^{\text{ret}}(x, y) \frac{\delta^2 S_{\text{tot}}}{\delta \varphi(y) \delta \varphi(z)} = \delta(x - z) = \int dy \frac{\delta^2 S_{\text{tot}}}{\delta \varphi(x) \delta \varphi(y)} \Delta_{S_{\text{tot}}}^{\text{ret}}(y, z) \quad (2.6)$$

and $\Delta_{S_{\text{tot}}}^{\text{ret}}(x, y) = 0$ for x sufficiently early. In the following we consider only actions S_{tot} for which the retarded Green function exists and is unique in the sense of formal power series in λ .

To introduce the perturbative expansion around the free theory and to define the Peierls bracket, we define retarded wave operators which map solutions of the free theory to solutions of the interacting theory [13]. However, we define them as maps on the space \mathcal{C} of all field configurations (*'off-shell formalism'*) and not only on the space of free solutions:

Definition 2.1. A retarded wave operator is a family of maps $(r_{S_0+S, S_0})_{S \in \mathcal{F}}$ from \mathcal{C} into itself with the properties

- (i) $r_{S_0+S, S_0}(f)(x) = f(x)$ for x sufficiently early
- (ii) $\frac{\delta(S_0+S)}{\delta \varphi} \circ r_{S_0+S, S_0} = \frac{\delta S_0}{\delta \varphi}$.

The following Lemma is proved in [6].

Lemma 2.2. *The retarded wave operator $(r_{S_0+S, S_0})_{S \in \mathcal{F}}$ exists and is unique and invertible in the sense of formal power series in the interaction S .*

Motivated by the interaction picture known from QFT, we introduce retarded fields: the classical retarded field to the interaction S and corresponding to the functional $F \in \mathcal{F}$ is defined by

$$F_S^{\text{cl}} \stackrel{\text{def}}{=} F \circ r_{S_0+S, S_0} : \mathcal{C} \longrightarrow \mathbb{C}. \quad (2.7)$$

The crucial factorization property,

$$(F \cdot G)_S^{\text{cl}} = F_S^{\text{cl}} \cdot G_S^{\text{cl}}, \quad (2.8)$$

cannot be maintained in the process of quantization, because quantum fields are distributions. This is why many proofs of symmetries in classical field theory do not apply to QFT (cf. Sect. 5).

The perturbative expansion around the free theory is defined by expanding the retarded fields with respect to the interaction. The coefficients are given by the classical retarded product R_{cl} [13]:

$$R_{\text{cl}} : \mathbb{T}\mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F}, \quad R_{\text{cl}}(S^{\otimes n}, F) \stackrel{\text{def}}{=} \left. \frac{d^n}{d\lambda^n} \right|_{\lambda=0} F \circ r_{S_0+\lambda S, S_0}, \quad (2.9)$$

where $\mathbb{TV} \stackrel{\text{def}}{=} \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{V}^{\otimes n}$ denotes the tensor algebra corresponding to some vector space \mathcal{V} . For non-diagonal entries, $R_{\text{cl}}(\otimes_{j=1}^n S_j, F)$ is determined by linearity and symmetry under permutations of S_1, \dots, S_n . Interacting fields can then be written as

$$F_S^{\text{cl}} \simeq \sum_{n=0}^{\infty} \frac{1}{n!} R_{\text{cl}}(S^{\otimes n}, F) \equiv R_{\text{cl}}(e_{\otimes}^S, F) . \quad (2.10)$$

The r.h.s. of \simeq is interpreted as a *formal power series* (i.e. we do not care about convergence of the series).

By means of the retarded wave operator one can define an off-shell version [6] of the Peierls bracket associated to the action S [24], $\{\cdot, \cdot\}_S : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F}$, and one verifies that this is indeed a Poisson bracket, i.e. that $\{\cdot, \cdot\}_S$ is linear, antisymmetric and satisfies the Leibniz rule and the Jacobi identity [13, 6].

Following [6], we are now going to derive the classical off-shell MWI from the factorization (2.8) and the definition of the retarded wave operators. Let \mathcal{J} be the ideal generated by the free field equation,

$$\mathcal{J} \stackrel{\text{def}}{=} \left\{ \sum_{n=1}^N \int dx_1 \dots dx_n \varphi(x_1) \dots \varphi(x_{n-1}) \frac{\delta S_0}{\delta \varphi(x_n)} f_n(x_1, \dots, x_n) \right\} \subset \mathcal{F} ,$$

with $N < \infty$ and the f_n 's being defined as in (2.2). Obviously, every $A \in \mathcal{J}$ can be written as

$$A \stackrel{\text{def}}{=} \int dx Q(x) \frac{\delta S_0}{\delta \varphi(x)} , \quad (2.11)$$

where Q may be non-local. Given $A \in \mathcal{J}$ we introduce a corresponding derivation [13]

$$\delta_A \stackrel{\text{def}}{=} \int dx Q(x) \frac{\delta}{\delta \varphi(x)} . \quad (2.12)$$

Notice $F(\varphi+Q) - F(\varphi) = \delta_A F + \mathcal{O}(Q^2)$ (for $F \in \mathcal{F}$) that is, $\delta_A F$ can be interpreted as the variation of F under the infinitesimal field transformation $\varphi(x) \mapsto \varphi(x) + Q(x)$. From the definition of the retarded wave operators Def. 2.1 we obtain

$$\begin{aligned} (A + \delta_A S) \circ r_{S_0+S, S_0} &= \int dx Q(x) \circ r_{S_0+S, S_0} \frac{\delta(S_0 + S)}{\delta \varphi(x)} \circ r_{S_0+S, S_0} \\ &= \int dx Q(x) \circ r_{S_0+S, S_0} \frac{\delta S_0}{\delta \varphi(x)} . \end{aligned} \quad (2.13)$$

In terms of the perturbative expansion this relation reads

$$R_{\text{cl}}(e_{\otimes}^S, A + \delta_A S) = \int dx R_{\text{cl}}(e_{\otimes}^S, Q(x)) \frac{\delta S_0}{\delta \varphi(x)} \in \mathcal{J} . \quad (2.14)$$

This is the MWI written in the off-shell formalism. When restricted to the solutions of the free field equation, the right-hand side vanishes and we obtain the on-shell

version of the MWI, as it was derived in [13]. For the simplest case $Q = 1$ the MWI reduces to the off-shell version of the (interacting) field equation

$$R_{\text{cl}}\left(e_{\otimes}^S, \frac{\delta(S_0 + S)}{\delta\varphi(x)}\right) = \frac{\delta S_0}{\delta\varphi(x)} . \quad (2.15)$$

3. Causal perturbation theory

Following [14], we quantize perturbative classical fields by deforming the underlying free theory as a function of \hbar : we replace \mathcal{F} by $\mathcal{F}[[\hbar]]$ (i.e. all functionals are formal power series in \hbar) and deform the classical product into the \star -product, $\star : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ (for simplicity we write \mathcal{F} for $\mathcal{F}[[\hbar]]$):

$$\begin{aligned} (F \star G)(\varphi) &\stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \int dx_1 \dots dx_n dy_1 \dots dy_n \frac{\delta^n F}{\delta\varphi(x_1) \dots \delta\varphi(x_n)} \\ &\quad \cdot \prod_{i=1}^n \Delta_m^+(x_i - y_i) \frac{\delta^n G}{\delta\varphi(y_1) \dots \delta\varphi(y_n)} . \end{aligned} \quad (3.1)$$

The \star -product is still associative but non-commutative.

In contrast to the classical retarded field F_S^{cl} (2.7), one assumes in perturbative QFT that the interaction S and the field F are *local* functionals. For an interacting quantum field F_S one makes the ansatz of a *formal power series* in the interaction S :

$$F_S = \sum_{n=0}^{\infty} \frac{1}{n!} R_{n,1}(S^{\otimes n}, F) \equiv R(e_{\otimes}^S, F) . \quad (3.2)$$

The 'retarded product' $R_{n,1}$ is a **linear** map, from $\mathcal{F}_{\text{loc}}^{\otimes n} \otimes \mathcal{F}_{\text{loc}}$ into \mathcal{F} which is **symmetric in the first n variables**. We interpret $R(A_1(x_1), \dots, A_n(x_n))$, $A_1, \dots, A_n \in \mathcal{P}$, as \mathcal{F} -valued distributions on $\mathcal{D}(\mathbb{M}^n)$, which are defined by: $\int dx h(x) R(\dots, A(x), \dots) := R(\dots \otimes A(h) \otimes \dots) \forall h \in \mathcal{D}(\mathbb{M})$.

Since the retarded products depend only on the functionals (and not on how the latter are written as smeared fields (2.4)), they must satisfy the **Action Ward Identity (AWI)** [14, 29, 30]:

$$\partial_\mu^x R_{n-1,1}(\dots A_k(x) \dots) = R_{n-1,1}(\dots, \partial_\mu A_k(x), \dots) . \quad (3.3)$$

Interacting fields are defined by the following axioms [14], which are motivated by their validity in classical field theory. The **basic axioms** are the initial condition $R_{0,1}(1, F) = F$ and

Causality: $F_{G+H} = F_G$ if $\text{supp}(\frac{\delta F}{\delta\varphi}) \cap (\text{supp}(\frac{\delta H}{\delta\varphi}) + \bar{V}_+) = \emptyset$ y;

GLZ Relation: $F_G \star H_G - H_G \star F_G = \frac{d}{d\lambda} \Big|_{\lambda=0} (F_{G+\lambda H} - H_{G+\lambda F})$.

Using only these requirements, the retarded products $R_{n,1}$ can be constructed by induction on n (cf. [28]). However, in each inductive step one is free to add a local functional, which corresponds to the usual renormalization ambiguity. This

ambiguity is reduced by imposing **renormalization conditions** as further axioms, see below.

Mostly, perturbative QFT is formulated in terms of the **time ordered product** (' T -product') $T : \mathbb{T}\mathcal{F}_{\text{loc}} \rightarrow \mathcal{F}$, which is a **linear** and **totally symmetric** map. Compared with the R -product, the T -product has the advantage of being totally symmetric and the disadvantage that its classical limit does not exist [11]. R - and T -products are related by Bogoliubov's formula:

$$R(e_{\otimes}^S, F) = \frac{\hbar}{i} \mathbf{S}(S)^{-1} \star \frac{d}{d\tau} \Big|_{\tau=0} \mathbf{S}(S + \tau F), \quad (3.4)$$

where

$$\mathbf{S}(S) \equiv T(e_{\otimes}^{iS/\hbar}) \equiv \sum_{n=0}^{\infty} \frac{i^n}{n! \hbar^n} T_n(S^{\otimes n}). \quad (3.5)$$

The basic axioms for retarded products translate into the following basic axioms for T -products: the initial conditions $T_0(1) = 1$, $T_1(F) = F$ and **causal factorization**:

$$T_n(A_1(x_1), \dots, A_n(x_n)) = T_k(A_1(x_1), \dots, A_k(x_k)) \star T_{n-k}(A_{k+1}(x_{k+1}), \dots, A_n(x_n)) \quad (3.6)$$

if $\{x_1, \dots, x_k\} \cap (\{x_{k+1}, \dots, x_n\} + \bar{V}_-) = \emptyset$. There is no axiom corresponding to the GLZ Relation. The latter can be interpreted as 'integrability condition' for the 'vector potential' $R(e_{\otimes}^S, F)$, that is it ensures the existence of the 'potential' $\mathbf{S}(S)$ fulfilling (3.4); for details see [7] and Proposition 2 in [10].

For this paper the following **renormalization conditions** are relevant (besides the MWI).

Translation Invariance: The group $(\mathbb{R}^d, +)$ of space and time translations has an obvious automorphic action β on \mathcal{F} , which is determined by $\beta_a \varphi(x) = \varphi(x + a)$, $a \in \mathbb{R}^d$. We require

$$\beta_a \mathbf{S}(S) = \mathbf{S}(\beta_a S), \quad \forall a \in \mathbb{R}^d. \quad (3.7)$$

Field Independence: $\frac{\delta T}{\delta \varphi(x)} = 0$. This axiom implies the causal Wick expansion of [15] as follows [14]: since $T(\otimes_{j=1}^n F_j) \in \mathcal{F}$ is polynomial in φ , it has a finite Taylor expansion in φ . By using Field Independence, this expansion can be written as

$$T_n(A_1(x_1), \dots, A_n(x_n)) = \sum_{l_1, \dots, l_n} \frac{1}{l_1! \dots l_n!} \cdot T_n \left(\dots, \sum_{a_{i1} \dots a_{il_i}} \frac{\partial^{l_i} A_i}{\partial(\partial^{a_{i1}} \varphi) \dots \partial(\partial^{a_{il_i}} \varphi)}(x_i), \dots \right) \Big|_{\varphi=0} \prod_{i=1}^n \prod_{j_i=1}^{l_i} \partial^{a_{ij_i}} \varphi(x_i) \quad (3.8)$$

with multi-indices $a_{ij_i} \in \mathbb{N}_0^d$.

Scaling: This requirement uses the *mass dimension of a monomial* in \mathcal{P} , which is defined by the conditions

$$\dim(\partial^a \varphi) = \frac{d-2}{2} + |a| \quad \text{and} \quad \dim(A_1 A_2) = \dim(A_1) + \dim(A_2) \quad (3.9)$$

for all monomials $A_1, A_2 \in \mathcal{P}$. The mass dimension of a *polynomial* in \mathcal{P} is the maximum of the mass dimensions of the contributing monomials. We denote by \mathcal{P}_{hom} the set of all field polynomials which are homogeneous in the mass dimension.

The axiom **Scaling Degree** requires that 'renormalization may not make the interacting fields more singular' (in the UV-region). Usually this is formulated in terms of Steinmann's *scaling degree* [28]:

$$\text{sd}(f) \stackrel{\text{def}}{=} \inf\{\delta \in \mathbb{R} \mid \lim_{\rho \downarrow 0} \rho^\delta f(\rho x) = 0\}, \quad f \in \mathcal{D}'(\mathbb{R}^k) \text{ or } f \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\}). \quad (3.10)$$

Namely, one requires

$$\text{sd}\left(T(A_1, \dots, A_n)|_{\varphi=0}(x_1 - x_n, \dots)\right) \leq \sum_{j=1}^n \dim(A_j), \quad \forall A_j \in \mathcal{P}_{\text{hom}}, \quad (3.11)$$

where Translation Invariance is assumed. Notice that this condition restricts *all* coefficients in the causal Wick expansion (3.8).

In the inductive construction of the sequence $(R_{n-1,1})_{n \in \mathbb{N}}$ or $(T_n)_{n \in \mathbb{N}}$, respectively, the problem of renormalization appears as the extension of the coefficients in the causal Wick expansion (which are $\mathbb{C}[[\hbar]]$ -valued distributions) from $\mathcal{D}(\mathbb{R}^{d(n-1)} \setminus \{0\})$ to $\mathcal{D}(\mathbb{R}^{d(n-1)})$. This extension has to be done in the sense of formal power series in \hbar , that is individually in each order in \hbar . With that it holds

$$\lim_{\hbar \rightarrow 0} R = R_{\text{cl}}. \quad (3.12)$$

In [14] it is shown that there **exists** a T -product which fulfills all axioms. The **non-uniqueness** of solutions is characterized by the 'Main Theorem'; for a complete version see [14].

4. Proper vertices

A main motivation for introducing proper vertices is to select that part of a T -product for which renormalization is non-trivial (cf. [21]). This is the contribution of all 1-particle-irreducible (1PI) subdiagrams. This selection can be done as follows: first one eliminates all disconnected diagrams. Then, one interprets each connected diagram as tree diagram with *non-local* vertices ('proper vertices') given by the 1PI-subdiagrams. The proper vertices can be interpreted as the 'quantum part' of the Feynman diagrams. Since renormalization is unique and trivial for tree diagrams, Ward identities can equivalently be formulated in terms of proper vertices (Sect. 5.1).

Essentially we follow this procedure, however, we avoid to argue in terms of diagrams, i.e. to use Wick's Theorem. It has been shown in [6] that with our definition (4.6) of the vertex functional Γ the 'proper interaction' $\Gamma(e_\otimes^S)$ corresponds to the sum of all 1PI-diagrams of $T(e_\otimes^{iS/\hbar})$.

The connected part T^c of a time-ordered T can be defined recursively by [11]

$$T_n^c(\otimes_{j=1}^n F_j) \stackrel{\text{def}}{=} T_n(\otimes_{j=1}^n F_j) - \sum_{|P| \geq 2} \prod_{J \in P} T_{|J|}^c(\otimes_{j \in J} F_j) . \quad (4.1)$$

It follows that T and T^c are related by the linked cluster theorem:

$$T(e_{\otimes}^{iF}) = \exp_{\bullet}(T^c(e_{\otimes}^{iF})) , \quad (4.2)$$

where \exp_{\bullet} denotes the exponential function with respect to the classical product.

For $F \in \mathcal{F}_{\text{loc}}$ the connected tree part $T_{\text{tree},n}^c(F^{\otimes n})$ can be defined as follows [11]: since $T_n^c = \mathcal{O}(\hbar^{n-1})$, the limit

$$\hbar^{-(n-1)} T_{\text{tree},n}^c \stackrel{\text{def}}{=} \lim_{\hbar \rightarrow 0} \hbar^{-(n-1)} T_n^c \quad (4.3)$$

exists. This definition reflects the well known statements that T_{tree}^c is the 'classical part' of T^c and that connected loop diagrams are of higher orders in \hbar .

Since proper vertices are non-local, we need the connected tree part $T_{\text{tree}}^c(\otimes_{j=1}^n F_j)$ for non-local entries $F_j \in \mathcal{F}$. This can be defined recursively [6]:

$$\begin{aligned} T_{\text{tree}}^c(\otimes_{j=1}^{n+1} F_j) &= \sum_{k=1}^n \int dx_1 \dots dx_k dy_1 \dots dy_k \frac{\delta^k F_{n+1}}{\delta \varphi(x_1) \dots \delta \varphi(x_k)} \\ &\cdot \prod_{j=1}^k \Delta_m^F(x_j - y_j) \frac{1}{k!} \sum_{I_1 \sqcup \dots \sqcup I_k = \{1, \dots, n\}} \frac{\delta}{\delta \varphi(y_1)} T_{\text{tree}}^c(\otimes_{j \in I_1} F_j) \cdot \dots \\ &\cdot \frac{\delta}{\delta \varphi(y_k)} T_{\text{tree}}^c(\otimes_{j \in I_k} F_j) , \end{aligned} \quad (4.4)$$

where $I_j \neq \emptyset \forall j$, \sqcup means the disjoint union and Δ_m^F is the Feynman propagator for mass m . (Note that in the sum over I_1, \dots, I_k the order of I_1, \dots, I_k is distinguished and, hence, there is a factor $\frac{1}{k!}$.) For local entries the two definitions (4.3) and (4.4) of T_{tree}^c agree, as explained in [6].

The 'vertex functional' Γ is defined by the following proposition [6]:

Proposition 4.1. *There exists a totally symmetric and linear map*

$$\Gamma : \mathbb{T}\mathcal{F}_{\text{loc}} \rightarrow \mathcal{F} \quad (4.5)$$

which is uniquely determined by

$$T^c(e_{\otimes}^{iS/\hbar}) = T_{\text{tree}}^c\left(e_{\otimes}^{i\Gamma(e_{\otimes}^S)/\hbar}\right) . \quad (4.6)$$

To zeroth and first order in S we obtain

$$\Gamma(1) = 0 , \quad \Gamma(S) = S . \quad (4.7)$$

Since T^c , T_{tree}^c and Γ are linear and totally symmetric, the defining relation (4.6) implies

$$T^c(e_{\otimes}^{iS/\hbar} \otimes F) = T_{\text{tree}}^c\left(e_{\otimes}^{i\Gamma(e_{\otimes}^S)/\hbar} \otimes \Gamma(e_{\otimes}^S \otimes F)\right) . \quad (4.8)$$

To prove the proposition, one constructs $\Gamma(\otimes_{j=1}^n F_j)$ by induction on n , using (4.6) and the requirements total symmetry and linearity:

$$\Gamma(\otimes_{j=1}^n F_j) = (i/\hbar)^{n-1} T^c(\otimes_{j=1}^n F_j) - \sum_{|P| \geq 2} (i/\hbar)^{|P|-1} T_{\text{tree}}^c \left(\bigotimes_{J \in P} \Gamma(\otimes_{j \in J} F_j) \right), \quad (4.9)$$

where P is a partition of $\{1, \dots, n\}$ in $|P|$ subsets J .

From this recursion relation and from $T_n^c - T_{\text{tree},n}^c = \mathcal{O}(\hbar^n)$ we inductively conclude

$$\Gamma(e_{\otimes}^S) = S + \mathcal{O}(\hbar), \quad \Gamma(e_{\otimes}^S \otimes F) = F + \mathcal{O}(\hbar) \quad \text{if } F, S \sim \hbar^0. \quad (4.10)$$

Motivated by this relation and (4.6) we call $\Gamma(e_{\otimes}^S)$ the 'proper interaction' corresponding to the classical interaction S .

The validity of renormalization conditions for T implies corresponding properties of Γ , as worked out in [6].

Analogously to the conventions for R - and T -products we sometimes write $\int dx g(x) \Gamma(A(x) \otimes F_2 \dots)$ for $\Gamma(A(g) \otimes F_2 \dots)$ ($A \in \mathcal{P}$, $g \in \mathcal{D}(\mathbb{M})$). Since Γ depends only on the *functionals*, it fulfills the AWI: $\partial_x^\mu \Gamma(A(x) \otimes F_2 \dots) = \Gamma(\partial^\mu A(x) \otimes F_2 \dots)$.

5. The Quantum Action Principle

5.1. Formulation of the Master Ward Identity in terms of proper vertices

The classical MWI was derived for arbitrary interaction $S \in \mathcal{F}$ and arbitrary $A \in \mathcal{J}$. For *local* functionals $S \in \mathcal{F}_{\text{loc}}$ and

$$A = \int dx h(x) Q(x) \frac{\delta S_0}{\delta \varphi(x)} \in \mathcal{J} \cap \mathcal{F}_{\text{loc}}, \quad h \in \mathcal{D}(\mathbb{M}), \quad Q \in \mathcal{P}, \quad (5.1)$$

it can be transferred formally into perturbative QFT (by the replacement $R_{\text{cl}} \rightarrow R$), where it serves as an additional, highly non-trivial renormalization condition:

$$R(e_{\otimes}^S, A + \delta_A S) = \int dy h(y) R(e_{\otimes}^S, Q(y)) \frac{\delta S_0}{\delta \varphi(y)}. \quad (5.2)$$

Since the MWI holds true in classical field theory (i.e. for connected tree diagrams, see below) it is possible to express this renormalization condition in terms of the 'quantum part' (described by the loop diagrams) - that is in terms of proper vertices. We do this in several steps:

Proof of the MWI for T_{tree}^c (connected tree diagrams). Since this is an alternative formulation of the *classical* MWI, we still include *non-local functionals* $S \in \mathcal{F}$, $A = \int dx Q(x) \frac{\delta S_0}{\delta \varphi(x)} \in \mathcal{J}$, as in Sect. 2. The classical field equation (2.15) can be expressed in terms of T_{tree}^c :

$$T_{\text{tree}}^c \left(e_{\otimes}^{iS/\hbar} \otimes \frac{\delta(S_0 + S)}{\delta \varphi(x)} \right) = \frac{\delta S_0}{\delta \varphi(x)}. \quad (5.3)$$

The only difference between R_{cl} and T_{tree}^c is that the retarded propagator $\Delta^{\text{ret}}(y) (\neq \Delta^{\text{ret}}(-y))$ is replaced by the Feynman propagator $\Delta^F(y) (= \Delta^F(-y))$, the combinatorics of the diagrams remains the same. Hence, the factorization of classical fields (2.8),

$$R_{\text{cl}}(e_{\otimes}^S, F \cdot G) = R_{\text{cl}}(e_{\otimes}^S, F) \cdot R_{\text{cl}}(e_{\otimes}^S, G) \quad (5.4)$$

holds true also for T_{tree}^c :

$$T_{\text{tree}}^c(e_{\otimes}^{iS/\hbar} \otimes FG) = T_{\text{tree}}^c(e_{\otimes}^{iS/\hbar} \otimes F) \cdot T_{\text{tree}}^c(e_{\otimes}^{iS/\hbar} \otimes G) . \quad (5.5)$$

We now multiply the field equation for T_{tree}^c with $T_{\text{tree}}^c(e_{\otimes}^{iS/\hbar} \otimes Q(x))$ and integrate over x . This yields the MWI for T_{tree}^c :

$$T_{\text{tree}}^c(e_{\otimes}^{iS/\hbar} \otimes (A + \delta_A S)) = \int dx T_{\text{tree}}^c(e_{\otimes}^{iS/\hbar} \otimes Q(x)) \cdot \frac{\delta S_0}{\delta \varphi(x)} . \quad (5.6)$$

Translation of the (quantum) MWI from R into T^c . Using Bogoliubov's formula (3.4) and the identity

$$(F \star G) \cdot \frac{\delta S_0}{\delta \varphi} = F \star (G \cdot \frac{\delta S_0}{\delta \varphi}) \quad \forall F, G \in \mathcal{F} \quad (5.7)$$

(which relies on $(\square + m^2)\Delta_m^+ = 0$), the MWI in terms of R -products (5.2) can be translated into T -products:

$$T(e_{\otimes}^{iS/\hbar} \otimes (A + \delta_A S)) = \int dy h(y) T(e_{\otimes}^{iS/\hbar} \otimes Q(y)) \frac{\delta S_0}{\delta \varphi(y)} , \quad h \in \mathcal{D}(\mathbb{M}) , Q \in \mathcal{P} . \quad (5.8)$$

To translate it further into T^c we note that the linked cluster formula (4.2) implies

$$T^c(e_{\otimes}^{iF} \otimes G) = T(e_{\otimes}^{iF})^{-1} \cdot T(e_{\otimes}^{iF} \otimes G) , \quad (5.9)$$

where the inverse is meant with respect to the classical product. It exists because $T(e_{\otimes}^{iF})$ is a formal power series of the form $T(e_{\otimes}^{iF}) = 1 + \mathcal{O}(F)$. With that we conclude that the MWI can equivalently be written in terms of T^c by replacing T by T^c on both sides of (5.8).

Translation of the MWI from T^c into Γ . Applying (4.6) on both sides of the MWI in terms of T^c we obtain

$$\begin{aligned} & \int dy h(y) T_{\text{tree}}^c(e_{\otimes}^{i\Gamma(e_{\otimes}^S)/\hbar} \otimes \Gamma(e_{\otimes}^S \otimes Q(y)) \frac{\delta(S_0 + S)}{\delta \varphi(y)}) \\ &= \int dy h(y) T_{\text{tree}}^c(e_{\otimes}^{i\Gamma(e_{\otimes}^S)/\hbar} \otimes \Gamma(e_{\otimes}^S \otimes Q(y))) \frac{\delta S_0}{\delta \varphi(y)} \\ &= \int dy h(y) T_{\text{tree}}^c(e_{\otimes}^{i\Gamma(e_{\otimes}^S)/\hbar} \otimes \Gamma(e_{\otimes}^S \otimes Q(y)) \frac{\delta(S_0 + \Gamma(e_{\otimes}^S))}{\delta \varphi(y)}) , \end{aligned}$$

where we have used the classical MWI in terms of T_{tree}^c (5.6). It follows

$$\Gamma(e_{\otimes}^S \otimes Q(y)) \frac{\delta(S_0 + \Gamma(e_{\otimes}^S))}{\delta \varphi(y)} = \Gamma(e_{\otimes}^S \otimes Q(y)) \frac{\delta(S_0 + S)}{\delta \varphi(y)} . \quad (5.10)$$

The various formulations of the MWI, in terms of R -products (5.2), T -products (5.8), T^c -products and in terms of proper vertices (5.10), they all are equivalent.

Remark 5.1. The off-shell field equation

$$T\left(e_{\otimes}^{iS/\hbar} \otimes \frac{\delta(S_0 + S)}{\delta\varphi(y)}\right) = \frac{\delta S_0}{\delta\varphi(y)} \cdot T\left(e_{\otimes}^{iS/\hbar}\right), \quad \forall S, \quad (5.11)$$

is a further renormalization condition, which can equivalently be expressed by

$$\Gamma(e_{\otimes}^S \otimes \varphi(y)) = \varphi(y), \quad \forall S, \quad (5.12)$$

as shown in [6]. For a T -product satisfying this condition and for $Q = D\varphi$ (where D is a polynomial in partial derivatives) the QAP simplifies to

$$D\varphi(y) \frac{\delta(S_0 + \Gamma(e_{\otimes}^S))}{\delta\varphi(y)} = \Gamma\left(e_{\otimes}^S \otimes D\varphi(y) \frac{\delta(S_0 + S)}{\delta\varphi(y)}\right). \quad (5.13)$$

5.2. The anomalous Master Ward Identity - Quantum Action Principle

The QAP is a statement about the structure of all possible violations of Ward identities. In our framework the main statement of the QAP is that any term violating the MWI can be expressed as $\Gamma(e_{\otimes}^S \otimes \Delta)$, where Δ is local (in a stronger sense than only $\Delta \in \mathcal{F}_{\text{loc}}$) and $\Delta = \mathcal{O}(\hbar)$ and the mass dimension of Δ is bounded in a suitable way.

Theorem 5.2 (Quantum Action Principle). *(a) Let Γ be the vertex functional belonging to a time ordered product satisfying the basic axioms and Translation Invariance (3.7). Then there exists a unique sequence of **linear** maps $(\Delta^n)_{n \in \mathbb{N}}$,*

$$\Delta^n : \mathcal{P}^{\otimes(n+1)} \rightarrow \mathcal{D}'(\mathbb{M}, \mathcal{F}_{\text{loc}}), \quad \otimes_{j=1}^n L_j \otimes Q \mapsto \Delta^n(\otimes_{j=1}^n L_j(x_j); Q(y)) \quad (5.14)$$

($\mathcal{D}'(\mathbb{M}, \mathcal{F}_{\text{loc}})$ is the space of \mathcal{F}_{loc} -valued distributions on $\mathcal{D}(\mathbb{M})$), which are symmetric in the first n factors,

$$\Delta^n(\otimes_{j=1}^n L_{\pi_j}(x_{\pi_j}); Q(y)) = \Delta^n(\otimes_{j=1}^n L_j(x_j); Q(y)) \quad (5.15)$$

for all permutations π , and which are implicitly defined by the 'anomalous MWI'

$$\Gamma(e_{\otimes}^S \otimes Q(y)) \frac{\delta(S_0 + \Gamma(e_{\otimes}^S))}{\delta\varphi(y)} = \Gamma\left(e_{\otimes}^S \otimes \left(Q(y) \frac{\delta(S_0 + S)}{\delta\varphi(y)} + \Delta(L; Q)(g; y)\right)\right), \quad (5.16)$$

where $S = L(g)$ ($L \in \mathcal{P}$, $g \in \mathcal{D}(\mathbb{M})$) and

$$\Delta(L; Q)(g; y) := \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n \prod_{j=1}^n g(x_j) \Delta^n(\otimes_{j=1}^n L(x_j); Q(y)). \quad (5.17)$$

As a consequence of (5.16) the maps Δ^n have the following properties:

- (i) $\Delta^0 = 0$;

- (ii) *locality*: there exist **linear** maps $P_a^n : \mathcal{P}^{\otimes(n+1)} \rightarrow \mathcal{P}$ (where a runs through a finite subset of $(\mathbb{N}_0^d)^n$), which are symmetric in the first n factors, such that Δ^n can be written as

$$\Delta^n(\otimes_{j=1}^n L_j(x_j); Q(y)) = \sum_{a \in (\mathbb{N}_0^d)^n} \partial^a \delta(x_1 - y, \dots, x_n - y) P_a^n(\otimes_{j=1}^n L_j; Q)(y) . \quad (5.18)$$

- (iii) $\Delta^n(\otimes_{j=1}^n L_j(x_j); Q(y)) = \mathcal{O}(\hbar) \quad \forall n > 0$ if $L_j \sim \hbar^0$, $Q \sim \hbar^0$.

(b) If the time ordered product satisfies the renormalization conditions Field Independence and Scaling Degree (3.11), then each term on the r.h.s. of (5.18) fulfills

$$|a| + \dim(P_a^n(\otimes_{j=1}^n L_j; Q)) \leq \sum_{j=1}^n \dim(L_j) + \dim(Q) + \frac{d+2}{2} - dn . \quad (5.19)$$

For a renormalizable interaction (that is $\dim(L) \leq d$) this implies

$$|a| + \dim(P_a^n(L^{\otimes n}; Q)) \leq \dim(Q) + \frac{d+2}{2} . \quad (5.20)$$

Note that (5.16) differs from the MWI (5.10) only by the local term $\Delta(L; Q)(g; y)$, which clearly depends on the chosen normalization of the time ordered product. Therefore, $\Delta(L; Q)(g; y) = 0$ is a sufficient condition for the validity of the MWI for Q and $S = L(g)$; it is also necessary due to the uniqueness of the maps Δ^n .

Proof. (a) Proceeding as in Sect. 5.1, the defining relation (5.16) can equivalently be written in terms of T -products:

$$T\left(e_{\otimes}^{iS/\hbar} \otimes (Q(y) \frac{\delta(S_0 + S)}{\delta\varphi(y)} + \Delta(L; Q)(g; y))\right) = T\left(e_{\otimes}^{iS/\hbar} \otimes Q(y)\right) \frac{\delta S_0}{\delta\varphi(y)} . \quad (5.21)$$

To n -th order in g this equation reads

$$\begin{aligned} \Delta^n(L^{\otimes n}; Q(y))(g^{\otimes n}) &= T\left((iS/\hbar)^{\otimes n} \otimes Q(y)\right) \cdot \frac{\delta S_0}{\delta\varphi(y)} - T\left((iS/\hbar)^{\otimes n} \otimes Q(y) \frac{\delta S_0}{\delta\varphi(y)}\right) \\ &- n T\left((iS/\hbar)^{\otimes n-1} \otimes Q(y) \frac{\delta S}{\delta\varphi(y)}\right) - \sum_{l=0}^{n-1} \binom{n}{l} T\left((iS/\hbar)^{\otimes n-l} \otimes \Delta^l(L^{\otimes l}; Q(y))(g^{\otimes l})\right) . \end{aligned} \quad (5.22)$$

Taking linearity and symmetry (5.15) into account we extend this relation to non-diagonal entries and write it in terms of the distributional kernels

$$\begin{aligned}
\Delta^n(\otimes_{j=1}^n L_j(x_j); Q(y)) &= \left(\frac{i}{\hbar}\right)^n T\left(\otimes_{j=1}^n L_j(x_j) \otimes Q(y)\right) \cdot \frac{\delta S_0}{\delta \varphi(y)} \\
&\quad - \left(\frac{i}{\hbar}\right)^n T\left(\otimes_{j=1}^n L_j(x_j) \otimes Q(y) \cdot \frac{\delta S_0}{\delta \varphi(y)}\right) \\
&\quad - \sum_{l=1}^n \left(\frac{i}{\hbar}\right)^{n-1} T\left(\otimes_{j(\neq l)} L_j(x_j) \otimes Q(y) \sum_a (\partial^a \delta)(x_l - y) \frac{\partial L_l}{\partial(\partial^a \varphi)}(x_l)\right) \\
&\quad - \sum_{I \subset \{1, \dots, n\}, I^c \neq \emptyset} \left(\frac{i}{\hbar}\right)^{|I^c|} T\left(\otimes_{i \in I^c} L_i(x_i) \otimes \Delta^{|I|}(\otimes_{j \in I} L_j(x_j); Q(y))\right) \quad (5.23)
\end{aligned}$$

This relation gives a unique inductive construction of the sequence $(\Delta^n)_{n \in \mathbb{N}}$ (if the distribution on the r.h.s. of (5.23) takes values in \mathcal{F}_{loc}) and it gives also the initial value $\Delta^0 = 0$. Obviously, the so obtained maps $\Delta^n : \mathcal{P}^{\otimes(n+1)} \rightarrow \mathcal{D}'(\mathbb{M}, \mathcal{F}_{\text{loc}})$ are linear and symmetric (5.15).

The main task is to prove that $\Delta^n(\otimes_{j=1}^n L_j; Q)$ (which is defined inductively by (5.23)) satisfies locality (5.18); the latter implies that $\Delta^n(\otimes_{j=1}^n L_j; Q)$ takes values in \mathcal{F}_{loc} . For this purpose we first prove

$$\text{supp } \Delta^n(\otimes_{j=1}^n L_j; Q) \subset \mathbb{D}_{n+1} \stackrel{\text{def}}{=} \{(x_1, \dots, x_{n+1}) \in \mathbb{M}^{n+1} \mid x_1 = \dots = x_{n+1}\}, \quad (5.24)$$

that is we show that the r.h.s. of (5.23) vanishes for $(x_1, \dots, x_n, y) \notin \mathbb{D}_{n+1}$. For such a configuration there exists a $K \subset \{1, \dots, n\}$ with $K^c := \{1, \dots, n\} \setminus K \neq \emptyset$ and either $(\{x_k \mid k \in K^c\} + \bar{V}_+) \cap (\{x_j \mid j \in K\} \cup \{y\}) = \emptyset$ or $(\{x_k \mid k \in K^c\} + \bar{V}_-) \cap (\{x_j \mid j \in K\} \cup \{y\}) = \emptyset$. We treat the first case, the second case is completely analogous. Using causal factorization of the T -products (3.6) and locality (5.24) of the inductively known $\Delta^{|I|}$, $|I| < n$, we write the r.h.s. of (5.23) as

$$\begin{aligned}
&\left(\frac{i}{\hbar}\right)^n \left(T\left(\otimes_{j \in K^c} L_j(x_j)\right) \star T\left(\otimes_{i \in K} L_i(x_i) \otimes Q(y)\right) \right) \frac{\delta S_0}{\delta \varphi(y)} \\
&\quad - T\left(\otimes_{j \in K^c} L_j(x_j)\right) \star \left[\left(\frac{i}{\hbar}\right)^n T\left(\otimes_{i \in K} L_i(x_i) \otimes Q(y) \frac{\delta S_0}{\delta \varphi(y)}\right) \right. \\
&\quad + \left(\frac{i}{\hbar}\right)^{n-1} \sum_{l \in K} T\left(\otimes_{i \in K, i \neq l} L_i(x_i) \otimes Q(y) \sum_a (\partial^a \delta)(x_l - y) \frac{\partial L_l}{\partial(\partial^a \varphi)}(x_l)\right) \\
&\quad \left. + \left(\frac{i}{\hbar}\right)^{|K^c|+|K \setminus I|} \sum_{I \subset K} T\left(\otimes_{i \in K \setminus I} L_i(x_i) \otimes \Delta^{|I|}(\otimes_{s \in I} L_s(x_s); Q(y))\right) \right]. \quad (5.25)
\end{aligned}$$

Using (5.7) this can be written in the form $T(\otimes_{j \in K^c} L_j(x_j)) \star (\dots)$. The second factor vanishes due to the validity of (5.23) in order $|K|$. This proves (5.24).

$\Delta^n(\otimes_{j=1}^n L_j; Q)$ is, according to its inductive definition (5.23), a distribution on $\mathcal{D}(\mathbb{M}^{n+1})$ which takes values in \mathcal{F} . Hence, it is of the form

$$\Delta^n(\otimes_{j=1}^n L_j(x_j); Q(y)) = \sum_k \int dz_1 \dots dz_k f_k^n(\otimes_{j=1}^n L_j \otimes Q)(x_1, \dots, x_n, y, z_1, \dots, z_k) \varphi(z_1) \dots \varphi(z_k) , \quad (5.26)$$

where $f_k^n(\otimes_{j=1}^n L_j \otimes Q)(x_1, \dots, x_n, y, z_1, \dots, z_k) \in \mathcal{D}'(\mathbb{M}^{n+k+1})$ has the following properties:

- it depends linearly on $(\otimes_{j=1}^n L_j \otimes Q)$;
- it is invariant under permutations of the pairs $(L_1, x_1), \dots, (L_n, x_n)$.
- The distribution $\int dx_1 \dots dx_n dy f_k^n(\otimes_{j=1}^n L_j \otimes Q)(x_1, \dots, x_n, y, z_1, \dots, z_k) h(x_1, \dots, x_n, y) \in \mathcal{D}'(\mathbb{M}^k)$ is symmetric under permutations of z_1, \dots, z_k and satisfies the wave front set condition (2.3), for all $h \in \mathcal{D}(\mathbb{M}^{n+1})$.
- From (5.23) we see that Translation Invariance of the T -product (3.7) implies the same property for Δ^n :

$$\beta_a \Delta^n(\otimes_{j=1}^n L_j(x_j); Q(y)) = \Delta^n(\otimes_{j=1}^n L_j(x_j + a); Q(y + a)) . \quad (5.27)$$

Therefore, the distributions $f_k^n(\otimes_{j=1}^n L_j \otimes Q)$ depend only on the relative coordinates.

Due to (5.24) the support of $f_k^n(\otimes_{j=1}^n L_j \otimes Q)$ is contained in $\mathbb{D}_{n+1} \times \mathbb{M}^k$; but, to obtain the assertion (5.18), we have to show $\text{supp } f_k^n(\otimes_{j=1}^n L_j \otimes Q) \subset \mathbb{D}_{n+k+1}$. For this purpose we take into account that

$$\frac{\delta T(\otimes_{j=1}^l A_j(x_j))}{\delta \varphi(z)} = 0 \quad \text{if } z \neq x_j \quad \forall j = 1, \dots, l . \quad (5.28)$$

This relation can be shown as follows: for the restriction of the time ordered product to $\mathcal{D}(\mathbb{M}^l \setminus \mathbb{D}_l)$ this property is obtained inductively by causal factorization (3.6). That (5.28) is maintained in the extension of the T -product to $\mathcal{D}(\mathbb{M}^l)$ can be derived from

$$[T(\otimes_{j=1}^l A_j(x_j)), \varphi(z)]_\star = 0 \quad \text{if } (x_j - z)^2 < 0 \quad \forall j = 1, \dots, l , \quad (5.29)$$

which is a consequence of the causal factorization of $T(\varphi(z) \otimes \otimes_{j=1}^l A_j(x_j))$ (cf. Sect. 3 of [15]).

Applying (5.28) to the T -products on the r.h.s. of (5.23) and using (5.24), we conclude

$$\text{supp } \frac{\delta \Delta^n(\otimes_{j=1}^n L_j; Q)}{\delta \varphi} \subset \mathbb{D}_{n+2} . \quad (5.30)$$

It follows that the distributions $f_k^n(\otimes_{j=1}^n L_j \otimes Q)$ (5.26) have support on the total diagonal \mathbb{D}_{n+k+1} . Taking additionally Translation Invariance into account, we conclude that these distributions are of the form

$$f_k^n(\otimes_{j=1}^n L_j \otimes Q)(x_1, \dots, x_n, y, z_1, \dots, z_k) = \sum_{a,b} C_{a,b}(\otimes_{j=1}^n L_j \otimes Q) \partial^a \delta(x_1 - y, \dots, x_n - y) \partial^b \delta(z_1 - y, \dots, z_k - y) , \quad (5.31)$$

where the coefficients $C_{a,b}(\otimes_{j=1}^n L_j \otimes Q) \in \mathbb{C}$ depend linearly on $(\otimes_{j=1}^n L_j \otimes Q)$ and are symmetric in the first n factors. Inserting (5.31) into (5.26) we obtain (5.18), the corresponding maps P_a^n having the asserted properties.

The important property (iii) is obtained by taking the classical limit $\hbar \rightarrow 0$ of the anomalous MWI (5.16): using (4.10) it results $\lim_{\hbar \rightarrow 0} \Delta(L; Q)(g; y) = 0$.

(b) The statement (5.19) is a modified version of Proposition 10(ii) in [6]. It follows from the formulas ([6]-5.32-33) and ([6]-5.46-47) of that paper. Namely, by using the causal Wick expansion of Δ^n (which follows from the Field Independence of the T -product) and (5.24) it is derived in ([6]-5.32-33) that Δ^n is of the form

$$\begin{aligned} \Delta^n(\otimes_{j=1}^n L_j(x_j); Q(y)) &= \sum_{\mathbf{l}, \mathbf{a}, b} C_{\mathbf{a}, b}^{\mathbf{l}} (\partial^b \delta)(x_1 - y, \dots, x_n - y) \\ &\quad \cdot \prod_{i=1}^n \prod_{j_i=1}^{l_i} \left(\partial^{a_{ij_i}} \varphi(x_i) \right) \cdot \prod_{j=1}^l \partial^{a_j} \varphi(y) \\ &= \sum_{\mathbf{l}, \mathbf{a}, b} \sum_{d \leq b} \tilde{C}_{\mathbf{a}, b, d}^{\mathbf{l}} (\partial^d \delta)(x_1 - y, \dots, x_n - y) \\ &\quad \cdot \prod_{i=1}^n \left(\partial^{b_i - d_i} \prod_{j_i=1}^{l_i} \left(\partial^{a_{ij_i}} \varphi(y) \right) \right) \cdot \prod_{j=1}^l \partial^{a_j} \varphi(y) , \end{aligned} \quad (5.32)$$

where $\mathbf{l} \equiv (l_1, \dots, l_n; l)$, $\mathbf{a} \equiv (a_{11}, \dots, a_{1l_1}, \dots, a_{n1}, \dots, a_{nl_n}; a_1 \dots a_l)$ and $C_{\mathbf{a}, b}^{\mathbf{l}}, \tilde{C}_{\mathbf{a}, b, d}^{\mathbf{l}}$ are numerical coefficients which depend also on (L_1, \dots, L_n, Q) . Since the T -product satisfies the axiom Scaling Degree the range of b is bounded by ([6]-5.46). The l.h.s. of (5.19) is given by

$$|d| + |b - d| + \sum_{i=1}^n \sum_{j_i=1}^{l_i} \left(|a_{ij_i}| + \frac{d-2}{2} \right) + \sum_{j=1}^l \left(|a_j| + \frac{d-2}{2} \right) , \quad (5.33)$$

which agrees with the l.h.s. of ([6]-5.47). Hence, it is bounded by the r.h.s. of ([6]-5.47). \square

Remark 5.3. Since the T -product $T(F^{\otimes n})$ depends only on the (local) functional F and not on how F is written as $F = \sum_k \int dx g_k(x) P_k(x)$ ($g_k \in \mathcal{D}(\mathbb{M})$, $P_k \in \mathcal{P}$), we conclude from (5.23) that we may express the violating term $\Delta(L; Q)(g; y)$ as follows: given $A = \int dx h(x) Q(x) \delta S_0 / \delta \varphi(x)$ ($h \in \mathcal{D}(\mathbb{M})$, $Q \in \mathcal{P}$), there exists a linear and symmetric map $\Delta_A : \mathbb{T}\mathcal{F}_{\text{loc}} \rightarrow \mathcal{F}_{\text{loc}}$ which is uniquely determined by

$$\Delta_A(e_{\otimes}^{L(g)}) \stackrel{\text{def}}{=} \int dy h(y) \Delta(L; Q)(g; y) . \quad (5.34)$$

A glance at (5.23) shows that Δ_A depends linearly on A . The corresponding smeared out version of the QAP is given in [6].

We are now going to reformulate our version of the QAP (Theorem 5.2) in the form given in the literature. Motivated by (4.10), we interpret $\Gamma_{\text{tot}}(S_0, S) \stackrel{\text{def}}{=} S_0 + \Gamma(e_{\otimes}^S)$ as the *proper total action* associated with the classical action $S_{\text{tot}} = S_0 + S$.

For $P \in \mathcal{C}^\infty(\mathbb{M}, \mathcal{P})$ the 'insertion' of $P(x)$ into $\Gamma_{\text{tot}}(S_0, S)$ is denoted and defined by⁴

$$P(x) \cdot \Gamma_{\text{tot}}(S_0, S) \stackrel{\text{def}}{=} \left. \frac{\delta}{\delta \rho(x)} \right|_{\rho \equiv 0} \Gamma_{\text{tot}}\left(S_0, S + \int dx \rho(x) P(x)\right) = \Gamma(e_\otimes^S \otimes P(x)) , \quad (5.35)$$

where $\rho \in \mathcal{D}(\mathbb{M})$ is an 'external field'. Setting $S' \stackrel{\text{def}}{=} S + \int dx \rho(x) Q(x)$ and introducing the local field

$$\Delta(x) \stackrel{\text{def}}{=} Q(x) \frac{\delta(S_0 + S)}{\delta \varphi(x)} + \Delta(L; Q)(g; x) \in \mathcal{C}^\infty(\mathbb{M}, \mathcal{P}) , \quad (5.36)$$

the anomalous MWI (5.16) can be rewritten as

$$\left. \frac{\delta \Gamma_{\text{tot}}(S_0, S')}{\delta \rho(x)} \frac{\delta \Gamma_{\text{tot}}(S_0, S')}{\delta \varphi(x)} \right|_{\rho \equiv 0} = \Delta(x) \cdot \Gamma_{\text{tot}}(S_0, S) . \quad (5.37)$$

The \hbar -expansion of the right-hand side starts with

$$\Delta(x) \cdot \Gamma_{\text{tot}}(S_0, S) = Q(x) \frac{\delta(S_0 + S)}{\delta \varphi(x)} + \mathcal{O}(\hbar) \equiv \left. \frac{\delta(S_0 + S')}{\delta \rho(x)} \frac{\delta(S_0 + S')}{\delta \varphi(x)} \right|_{\rho \equiv 0} + \mathcal{O}(\hbar), \quad (5.38)$$

where (4.10) is used. To discuss the mass dimension of the local insertion Δ (5.36), we assume that there is an open region $\emptyset \neq \mathcal{U} \subset \mathbb{M}$ such that the test function g which switches the interaction is constant in \mathcal{U} : $g|_{\mathcal{U}} = \text{constant}$. For $x \in \mathcal{U}$ the insertion $\Delta(x)$ is a field polynomial with constant coefficients. By $\dim(\Delta)$ we mean the mass dimension of this polynomial. For a renormalizable interaction Theorem 5.2(b) implies

$$\dim(\Delta) \leq \dim(Q) + \frac{d+2}{2} = \dim(Q) - \dim(\varphi) + d . \quad (5.39)$$

This version (5.37)-(5.39) of the QAP, which we have proved in the framework of CPT, formally agrees with the literature, namely with the 'QAP for nonlinear variations of the fields' (formulas (3.82)-(3.83) in [26]). This is the most important and most difficult case of the QAP.

As explained in (2.15), the MWI reduces for $Q = 1$ to the off-shell field equation. Setting $Q = 1$ in (5.37)-(5.39) and using $\Gamma(e_\otimes^S \otimes 1) = 1$, we obtain $\delta \Gamma_{\text{tot}}(S_0, S)/\delta \varphi(x) = \Delta(x) \cdot \Gamma_{\text{tot}}(S_0, S)$, where $\Delta(x) \cdot \Gamma_{\text{tot}}(S_0, S) = \delta(S_0 + S)/\delta \varphi(x) + \mathcal{O}(\hbar)$ and $\dim(\Delta) \leq d - \dim(\varphi)$, which formally agrees with formulas (3.80)-(3.81) in [26]. The latter are called there the 'QAP for the equations of motion', as expected from (2.15).

Remark 5.4. An 'insertion' (5.35) being a rather technical notion, the violating term $\Gamma(e_\otimes^S \otimes \Delta(L; Q)(g; y))$ in the anomalous MWI (5.16) can be much better interpreted by writing (5.16) in terms of R -products:

$$R\left(e_\otimes^S \otimes Q(y) \frac{\delta(S_0 + S)}{\delta \varphi(y)}\right) + R(e_\otimes^S \otimes \Delta(L; Q)(g; y)) = R(e_\otimes^S \otimes Q(y)) \frac{\delta S_0}{\delta \varphi(y)} . \quad (5.40)$$

⁴The dot does not mean the classical product here!

In this form, the violating term $R(e_{\otimes}^S \otimes \Delta(L; Q)(g; y))$ is the *interacting field* to the interaction S and belonging to the local field $\Delta(L; Q)(g; y)$.

6. Algebraic renormalization

In this section we sketch, for the *non-expert reader*, the crucial role of the QAP in algebraic renormalization. For shortness, we strongly simplify.

In algebraic renormalization one investigates, whether violations of Ward identities can be removed by finite renormalizations of the T -products. The results about the structure of the violating term given by the QAP are used as follows.

- Algebraic renormalization starts with the anomalous MWI (5.16), that is the result that the MWI can be violated only by an insertion term, i.e. a term of the form $\Gamma(e_{\otimes}^S \otimes \Delta)$ for some $\Delta \in \mathcal{F}_{\text{loc}}$, cf. (5.40).
- Algebraic renormalization proceeds by induction on the order of \hbar . To start the induction one uses that $\Delta \equiv \Delta(L; Q)(g; y)$ is of order $\mathcal{O}(\hbar)$.
- Because the finite renormalization terms, which one may add to a T -product, must be *local* (in the strong sense of (5.18)) and compatible with the axiom Scaling Degree, it is of crucial importance that $\Delta(L; Q)(g; y)$ satisfies locality (5.18) and the bound (5.19) on its mass dimension.

For many Ward identities it is possible to derive a *consistency equation* for $\Delta(L; Q)(g; y)$. Frequently this equation can be interpreted as the statement that $\Delta(L; Q)(g; y)$ is a *cocycle* in the cohomology generated by the corresponding symmetry transformation δ acting on some space $\mathcal{K} \subset \mathcal{F}_{\text{loc}}$. For example, δ is a nilpotent derivation (as the BRST-transformation⁵) or a family of derivations $(\delta_a)_{a=1, \dots, N}$ fulfilling a Lie algebra relation $[\delta_a, \delta_b] = f_{abc} \delta_c$.

If the cocycle $\Delta(L; Q)(g; y)$ is a *coboundary*, it is usually possible to remove this violating term by a finite renormalization. Hence, in this case, the solvability of the considered Ward identity amounts to the question whether this cohomology is trivial. For a renormalizable interaction the bound (5.19) on the mass dimension makes it possible to reduce the space \mathcal{K} to a finite dimensional space, this simplifies the cohomological question enormously.

Many examples for this pattern are given in [26]. In the framework of CPT the QAP and its application in algebraic renormalization have been used to prove the Ward identities of the $O(N)$ scalar field model [6] (as a simple example to illustrate how algebraic renormalization works in CPT) and, much more relevant, BRST-symmetry of Yang-Mills fields in curved space-times [20].

References

- [1] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in gauge theories*. Phys. Rept. **338** (2000), 439–569.

⁵The cohomological structure of BRST-symmetry is much richer as mentioned here, see [17].

- [2] C. Becchi, A. Rouet and R. Stora, *Renormalization of the abelian Higgs-Kibble Model*. Commun. Math. Phys. **42** (1975), 127–162
- [3] C. Becchi, A. Rouet and R. Stora, *Renormalization of gauge theories*. Annals Phys. **98** (1976), 287–321.
- [4] N.N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*. Interscience Publishers, Inc., New York (1959)
- [5] P. Breitenlohner and D. Maison, *Dimensional renormalization and the action principle*. Commun. Math. Phys. **52** (1977), 11–38
- [6] F. Brennecke and M. Dütsch, *Removal of violations of the Master Ward Identity in perturbative QFT*. Rev. Math. Phys. ... (2008), 100–120.
- [7] R. Brunetti, M. Dütsch and K. Fredenhagen, *Retarded products versus time-ordered products: a geometrical interpretation*, work in preparation
- [8] R. Brunetti and K. Fredenhagen, *Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds*. Commun. Math. Phys. **208** (2000), 623–661
- [9] M. Dütsch and F.-M. Boas, *The Master Ward identity*. Rev. Math. Phys. **14** (2002), 977–1049.
- [10] M. Dütsch and K. Fredenhagen, *A local (perturbative) construction of observables in gauge theories: The example of QED*. Commun. Math. Phys. **203** (1999), 71–105.
- [11] M. Dütsch and K. Fredenhagen, *Algebraic quantum field theory, perturbation theory, and the loop expansion*. Commun. Math. Phys. **219** (2001), 5–30.
- [12] M. Dütsch and K. Fredenhagen, *Perturbative algebraic field theory, and deformation quantization*. Fields Institute Communications **30** (2001), 151–160
- [13] M. Dütsch and K. Fredenhagen, *The Master Ward identity and generalized Schwinger-Dyson equation in classical field theory*. Commun. Math. Phys. **243** (2003), 275–314.
- [14] M. Dütsch and K. Fredenhagen, *Causal perturbation theory in terms of retarded products, and a proof of the Action Ward identity*. Rev. Math. Phys. **16** (2004), 1291–1348.
- [15] H. Epstein and V. Glaser, *The role of locality in perturbation theory*. Ann. Inst. H. Poincaré **A19** (1973), 211.
- [16] R. Haag, *Local Quantum Physics: Fields, Particles and Algebras*. 2nd edn., Springer-Verlag, Berlin (1996)
- [17] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*. Princeton University Press, Princeton, New Jersey (1992), 1–520.
- [18] S. Hollands and R.M. Wald, *Existence of local covariant time-ordered products of quantum fields in curved spacetimes*. Commun. Math. Phys. **231** (2002), 309–345
- [19] S. Hollands and R.M. Wald, *Conservation of the stress tensor in interacting quantum field theory in curved spacetimes*. Rev. Math. Phys. **17** (2005), 227–312
- [20] S. Hollands, *Renormalized Quantum Yang-Mills Fields in Curved Spacetime*. gr-qc/0705.3340 (2007)
- [21] G. Jona-Lasinio, *Relativistic Field Theories with Symmetry-Breaking Solutions*. Nuovo Cimento **34** (1964), 1790–1795

- [22] Y.-M. P. Lam, *Perturbation Lagrangian theory for scalar fields: Ward-Takahasi identity and current algebra*. Phys. Rev. **D6** (1972), 2145–2161
- [23] J. H. Lowenstein, *Differential vertex operations in Lagrangian field theory*. Commun. Math. Phys. **24** (1971), 1–21
- [24] R.E. Peierls, *The commutation laws of relativistic field theory*. Proc. Roy. Soc. (London) **A214** (1952), 143.
- [25] O. Piguet and A. Rouet, *Symmetries in perturbative quantum field theory*. Phys. Rept. **76** (1981), 1–77
- [26] O. Piguet and S. P. Sorella, *Algebraic renormalization: Perturbative renormalization, symmetries and anomalies*. Lect. Notes Phys. **M28** (1995), 1–134.
- [27] G. Pinter, *The Action Principle in Epstein Glaser Renormalization and Renormalization of the S Matrix of ϕ^4 -Theory*. hep-th/9911063
- [28] O. Steinmann, *Perturbative Expansion in axiomatic field theory*. Lect. Notes Phys. **11** (1971)
- [29] R. Stora, *Pedagogical Experiments in Renormalized Perturbation Theory*. Contribution to Conference 'Theory of Renormalization and Regularization', Hesselberg, Germany, <http://wwwthep.physik.uni-mainz.de/~scheck> (2002)
- [30] R. Stora, *The Wess Zumino consistency condition: a paradigm in renormalized perturbation theory*. Fortsch. Phys. **54** (2006), 175–182
- [31] I. V. Tyutin, *Gauge invariance in field theory and statistical physics in operator formalism*. LEBEDEV-75-39
- [32] W. Zimmermann, Commun. Math. Phys. **15** (1969), 208

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